SCATTERING POLES FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. For a class of manifolds X that includes quotients of real hyperbolic $(n+1)$-dimensional space by a convex co-compact discrete group, we show that the resonances of the meromorphically continued resolvent kernel for the Laplacian on X coincide, with multiplicities, with the poles of the meromorphically continued scattering operator for X. In order to carry out the proof, we use Shmuel Agmon's perturbation theory of resonances to show that both resolvent resonances and scattering poles are simple for generic potential perturbations.

1. Introduction

The purpose of this paper is to show the equivalence of two possible notions of 'scattering resonances' for the Laplacian on asymptotically hyperbolic manifolds, i.e., complete Riemannian manifolds of infinite volume with 'constant curvature at infinity'. On the one hand, it is very natural to define scattering resonances with respect to the meromorphically continued resolvent of the Laplace operator. This point of view has been very fruitful and has led to a large body of results on the distribution of scattering resonances in the complex plane; see [40] for a survey and [5, 6, 7, 41] for results on the class of manifolds studied here. On the other hand, it is also reasonable, by analogy with scattering theory for the Schrödinger or wave equations in Euclidean space (see e.g. [15, 36]), to define scattering resonances as poles of the scattering operator for the Laplacian. For Schrödinger scattering on Euclidean space, the equivalence of scattering resonances and resolvent resonances is well-known [8, 10, 11]. The analogous result for hyperbolic manifolds is of interest since the poles of the scattering operator have a geometric and dynamical interpretation: they are among the poles of Selberg's zeta function for geodesic flow on the manifold [29, 30, 32]. Thus the scattering resonances serve, in a sense, as discrete data similar in character to the eigenvalues of a compact surface.

For non-compact Riemann surfaces and certain metric perturbations, Guillopé and Zworski showed that the set of resolvent resonances and the set of scattering poles coincide with multiplicities ([7], Proposition 2.11); here we will show that, with some restrictions, the same equivalence holds for asymptotically hyperbolic manifolds in higher dimension.

To describe our results, we first recall that an asymptotically hyperbolic manifold is a compact manifold $\overline{X}$ with boundary endowed with a Riemannian metric

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of a special form. A *defining function* for the boundary of a compact manifold $\overline{X}$ is a nonnegative C^∞ function on $\overline{X}$ with $x^{-1}(0) = \partial\overline{X}$ and $dx|_{\partial\overline{X}}$ nowhere vanishing. The metric g then takes the form $x^{-2}h$ where x is a defining function and h is a nondegenerate smooth metric on $\overline{X}$ such that $|dx|_h \to 1$ as $x \to 0$. Note that this metric puts $\partial\overline{X}$ 'at infinity' and makes X, the interior of $\overline{X}$, a complete Riemannian manifold of infinite volume. The condition on $|dx|_h$ insures that the sectional curvatures approach -1 at metric infinity. If Δ_g denotes the positive Laplace-Beltrami operator on (X, g) and X has dimension $n+1$, it is known that the spectrum of Δ_g consists of at most finitely many L^2 eigenvalues of finite multiplicity in the interval $[0, n^2/4)$ (see [16] for quotients of hyperbolic space and [21] for asymptotically hyperbolic manifolds) and absolutely continuous spectrum in $[n^2/4, \infty)$ (see [3, 17] for hyperbolic quotients and [22] for resolvent estimates which imply absolute continuity of the spectrum for asymptotically hyperbolic manifolds). Thus the resolvent $\mathcal{R}_g(z) = (\Delta_g - z)^{-1}$ is a meromorphic operator-valued function on the cut plane $\mathbb{C}\backslash[n^2/4, \infty)$ with poles at the L^2 eigenvalues having finite-rank residues. It is convenient to introduce a uniformizing parameter ζ and set $R_g(\zeta) = (\Delta_g - \zeta(n-\zeta))^{-1}$, which is then a meromorphic operator-valued function on the half-plane $\Re(\zeta) > n/2$. The operator $R_g(\zeta)$ has first-order poles at points ζ_0 whenever $\zeta_0(n-\zeta_0)$ is an $L^2(X)$-eigenvalue of Δ_g. We denote by Z_p the (finite and possibly empty) set of all such ζ_0. The *multiplicity* of $\zeta_0 \in Z_p$ is the dimension, m_{ζ_0}, of the eigenspace of Δ_g with eigenvalue $\zeta_0(n-\zeta_0)$. Equivalently,

$$m_{\zeta_0} = \operatorname{rank}\left(\int_{\gamma_{\zeta_0}} R_g(\zeta)\, d\zeta\right), \tag{1.1}$$

where γ_{ζ_0} is a simple closed contour surrounding ζ_0 and no other pole of $R_g(\zeta)$.

First, we define the resolvent resonance set of Δ_g. Let $\dot{C}^\infty(X)$ denote the smooth functions on $\overline{X}$ vanishing to all orders at $\partial\overline{X}$. Viewed as a map from $\dot{C}^\infty(X)$ to $C^\infty(X)$, the resolvent operator $R_g(\zeta)$ admits a meromorphic continuation to $\mathbb{C} - \frac{1}{2}(n - \mathbb{N})$, as was shown in [22]. Resolvent resonances are poles of this meromorphic continuation in the half-plane $\Re(\zeta) < n/2$, excluding the region $\frac{1}{2}(n-\mathbb{N})$. If X has constant curvature in a neighborhood of infinity, the analysis of [6] shows that the resolvent has a meromorphic continuation to $\Re(\zeta) < n/2$ with no further restriction. Lower bounds on resolvent resonances proven in [38] show that this set is always nontrivial for constant curvature spaces, and explicit examples (see, for instance, section 3 of [5]) show that resolvent resonances can form an infinite lattice in the half-plane $\Re(\zeta) < n/2$. If ζ_0 is a resolvent resonance and $\Re(\zeta_0) < n/2$, the *multiplicity* of ζ_0 is the number

$$m_{\zeta_0} = \operatorname{rank}\left(\int_{\gamma_{\zeta_0}} R_g(\zeta)\, d\zeta\right) \tag{1.2}$$

(cf. (1.1)). Here γ_{ζ_0} is a simple closed curve that encloses ζ_0 and no other pole of $R_g(\zeta)$. The point ζ_0 is a *semi-simple resonance* if $R_g(\zeta)$ has a simple pole at ζ_0. The point ζ_0 is a *simple resonance* if, in addition, the residue of the pole has rank one. The *resolvent resonance set* is the set $\mathcal{R}$ of all ζ_0 such that $\Re(\zeta_0) < n/2$, $\zeta_0 \notin \frac{1}{2}(n-\mathbb{N})$, and $R_g(\zeta)$ has a pole of multiplicity $m_{\zeta_0} \neq 0$ at ζ_0.

Next, we define the scattering operator and the scattering resonance set of Δ_g. For $\zeta \in \mathbb{C}$ with $\Re(\zeta) = n/2$ and $\zeta \neq n/2$, and each $f_- \in C^\infty(\partial\overline{X})$, there is a unique smooth solution of the eigenvalue equation $(\Delta_g - \zeta(n-\zeta))u = 0$ having the

asymptotic form

$$u = x^{n-\zeta} f_+ + x^{\zeta} f_- + O(x^{n/2+1})$$

where $f_+ \in C^\infty(\partial\overline{X})$; for a proof see [2] or [12]. It follows that f_+ is uniquely determined and that there is a linear map $S(\zeta) : C^\infty(\partial\overline{X}) \to C^\infty(\partial\overline{X})$ with $S(\zeta) f_- = f_+$; moreover it is clear that $S(\zeta)S(n-\zeta) = I$. It can be shown that $S(\zeta)$ extends to a meromorphic family of operators on $\mathbb{C}$ (see [2] or [12]); these operators may have infinite-rank poles at $\zeta \in n/2 + \mathbb{N}$, and infinite-rank zeros at $\zeta \in n/2 - \mathbb{N}$. A scattering pole is a pole of the meromorphic continuation of $S(\zeta)$ in the half-plane $\Re(\zeta) < n/2$, excluding the set $\frac{1}{2}(n - \mathbb{N})$ unless X has constant curvature near infinity. It can be shown that the continued operator admits the factorization

$$S(\zeta) = P(\zeta)(I + K(\zeta))Q(\zeta) \tag{1.3}$$

holds, where $P(\zeta)$ and $Q(\zeta)$ are holomorphically invertible families of elliptic operators for $\zeta \in \mathbb{C} - \frac{1}{2}(n - \mathbb{N})$ and $K(\zeta)$ is a meromorphic family of compact operators on $C^\infty(\partial\overline{X})$. For $\zeta_0 \notin \frac{1}{2}(n - \mathbb{N})$, the *multiplicity* of a scattering pole ζ_0 is the integer

$$\nu_{\zeta_0} = \frac{1}{2\pi i} \operatorname{Tr} \left(\int_{\gamma_{\zeta_0,\varepsilon}} S(\zeta)^{-1} S'(\zeta)\, d\zeta \right) \tag{1.4}$$

(compare [7, 30] where similar definitions are made). The factorization (1.3) and results of [4] show that ν_{ζ_0} is an integer equal to the multiplicity of zeros minus the multiplicity of poles of $(I + K(\zeta))$ at $\zeta = \zeta_0$ (we give a precise formulation in section 3). We will say that ζ_0 is semi-simple if the pole of $S(\zeta)$ is of first order, and simple if the residue is rank-one. The *scattering resonance set* is the set $\mathcal{S}$ of all pairs $(\zeta_0, -\nu_{\zeta_0})$ with $\Re(\zeta_0) < n/2$, $\zeta \notin \frac{1}{2}(n - \mathbb{N})$, and $\nu_{\zeta_0} \neq 0$.

We would like to show a correspondence, with multiplicities, between the sets $\mathcal{R}$ and $\mathcal{S}$. A direct method (see for example [7], where the case $n = 1$ is treated) would compare the Laurent expansion of the meromorphically continued resolvent at $\zeta_0 \in \mathcal{R}$ to the Laurent expansion of the scattering operator at ζ_0, using the fact that the Schwarz kernel of the scattering operator can be recovered from that of the resolvent kernel. This direct method works easily when the resolvent resonance is simple but is somewhat complicated for non-simple resonances. For this reason, we will perturb the operator Δ_g with a potential $V \in \dot{C}^\infty(X)$ which, as we will show, can be chosen to make all resonances of the meromorphically continued resolvent

$$R_V(\zeta) = (\Delta_g + V - \zeta(n-\zeta))^{-1}$$

simple. The perturbation will split each resonance of multiplicity m into m resonances of multiplicity one localized near the unperturbed resonance, and similarly each eigenvalue of multiplicity m into m eigenvalues of multiplicity one. This result, which is of some independent interest, will allow us to count multiplicities properly but avoid technicalities associated with non-simple resonances.

Our analysis of generic potential perturbations is inspired by Klopp and Zworski's analysis of resonances in potential scattering [14]. To carry out the analysis, we will use Shmuel Agmon's perturbation theory of resonances [1] in which the resonances are realized as eigenvalues of a non-self-adjoint operator on a cleverly constructed Banach space; this replaces the complex scaling used in [14]. Standard Kato-Rellich perturbation theory [13] can then be used to study how the resonances move under perturbation.

Our first result is:

Theorem 1.1. *Let (X, g) be an asymptotically hyperbolic manifold, and let $\mathcal{R}$ and $\mathcal{S}$ be respectively the resolvent resonance set and scattering resonance set for the Laplacian Δ_g. Then $\mathcal{R} = \mathcal{S}$ on $\mathbb{C} - \frac{1}{2}(n - \mathbb{N})$ except for at most finitely many points. More precisely, for any ζ_0 with $\Re(\zeta_0) < n/2$, $\zeta_0 \notin \frac{1}{2}(n - \mathbb{N})$, the relationship*

$$\nu_{\zeta_0} = m_{n-\zeta_0} - m_{\zeta_0}$$

holds.

Note that $m_{n-\zeta_0}$ is nonzero only for the finitely many ζ_0 with $n - \zeta_0 \in Z_p$; for all other $\zeta_0 \in \frac{1}{2}(n - \mathbb{N})$, the scattering resonances and resolvent resonances coincide with multiplicities.

We can make a stronger statement if (X, g) has even dimension and constant curvature in a neighborhood of infinity, i.e., if there is a compact subset K of X so that g has constant negative curvature -1 on $X \backslash K$. This class was studied in [6] and includes convex co-compact hyperbolic manifolds. Recall that a geometrically finite group Γ of isometries of real hyperbolic $(n+1)$–dimensional space $\mathbb{H}^{n+1}$ is called convex co-compact if the orbit space $\Gamma \backslash \mathbb{H}^{n+1}$ has infinite volume and Γ contains no parabolic elements. If Γ is torsion-free (which we can insure by passing to a subgroup of finite index), the orbit space $X = \Gamma \backslash \mathbb{H}^{n+1}$ is a complete Riemannian manifold when given the induced hyperbolic metric g.

To formulate a result, we introduce a renormalized scattering operator

$$S_r(\zeta) = \frac{\Gamma(s - n/2)}{\Gamma(n/2 - s)} S(\zeta);$$

if $\dim(X)$ is even (so n is odd), the renormalization has the effect of dividing out the infinite-rank zeros of $S(\zeta)$ at $\zeta \in n/2 - \mathbb{N}$. Note, however, that it may also cancel poles of the scattering operator due to eigenvalues of Δ_g if the set $Z_p \cap (\frac{1}{2}n + \mathbb{N})$ is nonempty. It can be shown that the factorization

$$S_r(\zeta) = P(\zeta)(I + K(\zeta))P(\zeta)$$

holds for a family of operators $P(\zeta)$ holomorphically invertible in all of $\mathbb{C}$, and a meromorphic family of compact operators $K(\zeta)$ with at most finite rank zeros and poles. We then define the multiplicity of a scattering pole using (1.4) but with $S(\zeta)$ replaced by $S_r(\zeta)$ in the definition, and enlarge the set $\mathcal{S}$ to include any ζ_0 with $\Re(\zeta_0) < n/2$ where $S_r(\zeta)$ has a pole. Similarly, we enlarge the set $\mathcal{R}$ to include all resolvent poles ζ with $\Re(\zeta) < n/2$. With these definitions, we have:

Theorem 1.2. *Let (X, g) have constant curvature near infinity and suppose that* $\dim(X)$ *is even. Let $\mathcal{R}$ and $\mathcal{S}$ denote the resolvent resonance and scattering resonance sets for Δ_g. Then the relation*

$$\nu_{\zeta_0} = m_{n-\zeta_0} - m_{\zeta_0}$$

holds for all ζ_0 with $n - \zeta_0 \notin Z_p \cap (\frac{1}{2}n + \mathbb{N})$.

Thus, for all but finitely many ζ_0, the resolvent resonance set and the scattering resonance set coincide with multiplicities. The set $Z_p \cap \left(\frac{1}{2}n + \mathbb{N}\right)$ consists at most of finitely many elements, and is empty if $n = 1$.

This paper is organized as follows. In section 2 we review the Mazzeo-Melrose construction of the resolvent and study its behavior near resolvent resonances. In section 3 we recall how the scattering operator can be recovered from the resolvent

and discuss its behavior near scattering poles. In section 4 we study the perturbation behavior of resonances when the operator Δ_g is perturbed by a potential $V \in \dot{C}^\infty(X)$. In section 5, we show that the operator $P_V = \Delta_g + V$ has only simple resonances ζ_0 for $\zeta_0 \in \frac{1}{2}(n - \mathbb{N})$ for potentials V in a dense open subset of $\dot{C}^\infty(X)$. Finally, in section 6, we prove Theorems 1.1 and 1.2.

In what follows, $x^N L^2(X)$ denotes the space of locally square-integrable functions v on X with $v = x^N u$ for a function $u \in L^2(X)$ and a fixed real number N. For a fixed, given N, we denote by B_0 the Banach space $x^N L^2(X)$, and by B_1 the Banach space $x^{-N} L^2(X)$. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. We metrize $\dot{C}^\infty(X)$ by introducing the seminorms

$$d_{\alpha,n}(V) = \sup_{x \in X} \left| x^{-n} D^\alpha V(x) \right| \tag{1.5}$$

for nonnegative integers n and multi-indices α, and we denote by $d(\,\cdot\,,\,\cdot\,)$ the associated metric on $\dot{C}^\infty(X)$.

2. The Resolvent of $\Delta_g + V$ and its Meromorphic Continuation

The resolvent of the operator $P_V = \Delta_g + V$ has a distribution kernel, with respect to the Riemannian density on X, which is smooth away from the diagonal Λ of $X \times X$. To describe its singularities, it is useful to introduce the blow up of $\overline{X} \times \overline{X}$ along $\partial\overline{X} \times \partial\overline{X}$, the stretched product $\overline{X} \times_0 \overline{X}$. This amounts to introducing polar coordinates at the diagonal in the corner of $\overline{X} \times \overline{X}$ where Λ intersects the 'top' boundary face $\overline{X} \times \partial\overline{X}$ and the 'bottom' boundary face $\partial\overline{X} \times \overline{X}$; globally one replaces $\partial\Lambda$ with the doubly inward-pointing spherical normal bundle of $\partial\Lambda$. If (x, y) and (x', y') are local coordinates on $\overline{X}$ in a neighborhood of the boundary, $\partial\Lambda$ is given by $x = x' = y - y' = 0$; local coordinates for $\overline{X} \times_0 \overline{X}$ near the boundary are then given by $(r, \eta, \eta', \theta, y)$ where

$$r = \sqrt{x^2 + (x')^2 + |y - y'|^2}$$

$$(\eta, \eta', \theta) = (x/r, x'/r, (y - y')/r).$$

We denote by β the 'blow-down map' $\beta : \overline{X} \times_0 \overline{X} \to \overline{X} \times \overline{X}$; in the local coordinates described above, $\beta(r, \eta, \eta', \theta, y) = (r\eta, y, r\eta', y - r\theta)$.

The following theorem summarizes the Mazzeo-Melrose [22] construction of the resolvent. Although Mazzeo and Melrsoe did not treat potential perturbations, potentials in the class $\dot{C}^\infty(X)$ may be accommodated without difficulty (see [12], Theorem 3.1 and its proof). We denote by G_ζ the integral kernel of the resolvent operator $R_V(\zeta) = (P_V - \zeta(n - \zeta))^{-1}$ with respect to Riemannian measure on X, initially a meromorphic function of ζ with $\Re(\zeta) > n/2$.

Theorem 2.1. *Let (X, g) be an asymptotically hyperbolic manifold, and let $V \in \dot{C}^\infty(X)$. The resolvent kernel G_ζ has a meromorphic continuation to $\mathbb{C}$ with*

$$\beta^* G_\zeta = A_\zeta + B_\zeta + C_\zeta$$

where

$$A_\zeta \in I^{-2}(\overline{X} \times_0 \overline{X}),$$

$$B_\zeta \in (\eta\eta')^\zeta C^\infty(\overline{X} \times_0 \overline{X}),$$

and

$$C_\zeta \in \beta^* \left[(xx')^\zeta C^\infty(\overline{X} \times \overline{X}) \right].$$

Moreover A_ζ *is an entire function of* ζ*,* B_ζ *is holomorphic in* $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$*, and* C_ζ *is meromorphic in* $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$*.*

Sketch of proof: Let $P_\zeta = \Delta_g + V - \zeta(n-\zeta)$. Given an operator A on $C^\infty(X)$, we will denote by $\kappa(A)$ the lift of the kernel of A (with respect to the Riemannian density on X) to $X \times_0 X$. The construction in [22] may be broken into three pieces. First, we construct an operator A_ζ to cancel the conormal singularity of $\kappa(P_\zeta)$ on the lifted diagonal. The family A_ζ is entire and has the property that $\kappa(I - P_\zeta A_\zeta) \in C^\infty(\overline{X} \times_0 \overline{X})$. This remainder does not yet correspond to the integral kernel of a compact operator on the original space.

To improve the error term, one uses the model resolvent. A second operator B_ζ is constructed so that $E_\zeta = I - P_\zeta(A_\zeta + B_\zeta)$ has

$$\kappa(E_\zeta) \in \eta^\zeta (\eta' r)^\infty C^\infty(\overline{X} \times_0 \overline{X}).$$

The operator B_ζ is holomorphic in $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$; the operator E_ζ is a compact operator on the weighted L^2 space $x^N L^2(X)$ for all ζ with $\Re(\zeta) > n/2 - N$, and is also holomorphic in $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$.

Finally, one inverts $(I - E_\zeta)$ using analytic Fredholm theory. Composition theorems of [20] show that if

$$(I + F_\zeta) = (I - E_\zeta)^{-1},$$

then $\kappa(F_\zeta)$ also lies in $\eta^\zeta(\eta' r)^\infty C^\infty(X \times_0 X)$. This in turn implies that $C_\zeta = (A_\zeta + B_\zeta)F_\zeta$ belongs to $\beta^* \left[(xx')^\zeta C^\infty(\overline{X} \times \overline{X}) \right]$, and therefore $\beta^* G_\zeta$ has the claimed form. □

Remark 2.2. It follows from the form of the resolvent kernel that $R_V(\zeta)$ is a continuous mapping from $\dot{C}^\infty(X)$ to $C^\infty(X)$ when defined, and extends to a bounded mapping from $x^N L^2(X)$ to $x^{-N} L^2(X)$ for $\Re(\zeta) > n/2 - N$.

Remark 2.3. The Mazzeo-Melrose construction does not rule out the possibility of poles at $\zeta \in \frac{1}{2}(n-\mathbb{N})$, possibly of infinite rank. If (X,g) has constant negative curvature in a neighborhood of infinity, the operator $A_\zeta + B_\zeta$ may be replaced by the model resolvent (the resolvent of the Laplacian on the covering space $\mathbb{H}^{n+1}$), which is entire if n is even and has finite-rank poles at $\zeta = -k$ if n is odd (see for example the explicit formulas in [6], section 2) . In either case, these terms contain only poles of finite rank, and the last step of the construction, involving the meromorphic Fredholm theorem, gives at most poles with finite-rank residues. A detailed construction of the resolvent in this case is given in [6], section 3. This observation plays a crucial role in the proof of Theorem 1.2.

Theorem 2.1 and standard arguments (see [7] , Lemma 2.4) enable us to characterize the polar part of $R_V(\zeta)$ at a resolvent resonance $\zeta_0 \notin \frac{1}{2}(n-\mathbb{N})$. We will view the meromorphically continued resolvent as a mapping from the space $B_0 = x^N L^2(X)$ to $B_1 = x^{-N} L^2(X)$ as in Remark 2.2, where N is chosen so that $\Re(\zeta_0) > n/2 - N$. Introduce the nondegenerate form

$$\langle u, v \rangle = \int_X uv\, dx$$

(no complex conjugation) which can be used to pair elements in B_0 and B_1. The resolvent operator is symmetric with respect to this form.

Proposition 2.4. *Let $\zeta_0 \in \mathbb{C} - \frac{1}{2}(n - \mathbb{N})$ be a pole of $R_V(\zeta)$. Then*

$$R_V(\zeta) = \sum_{j=-k}^{-1} (\zeta(n-\zeta) - \zeta_0(n-\zeta_0))^j \, A_j + H(\zeta)$$

where $H(\zeta)$ is a holomorphic $\mathcal{L}(B_0, B_1)$-valued function near $\zeta = \zeta_0$ and the A_j are finite-rank operators in $\mathcal{L}(B_0, B_1)$ with

$$A_{-j} = (\Delta_g + V - \zeta_0(n-\zeta_0))^{j-1} A_{-1}$$

for $j \geq 2$. The operator A_{-1} commutes with $\Delta_g + V - \zeta_0(n-\zeta_0)$. Moreover, there is a basis $\{\psi_i\}_{i=1}^{m_{\zeta_0}}$ for Ran (A_{-1}) *so that*

$$A_{-1} f = \sum_i \langle f, \psi_i \rangle \; \psi_i,$$

and the operator $(\Delta_g + V - \zeta_0(n-\zeta_0))$ is represented on Ran (A_{-1}) *by a matrix M with $M^{k-1} \neq 0$ but $M^k = 0$.*

Remark 2.5. If (X, g) has constant curvature in a neighborhood of infinity, the same result holds for any resolvent resonance ζ_0 including those $\zeta_0 \in \frac{1}{2}(n - \mathbb{N})$.

3. The Scattering Operator for P_V

Let $S_V(\zeta)$ denote the scattering operator for $P_V = \Delta_g + V$. To describe the scattering operator and its singularities, we recall its connection with the resolvent kernel. First, we blow up $\partial\overline{X} \times \partial\overline{X}$ along the diagonal Λ_∞ to obtain the space $\partial\overline{X} \times_0 \partial\overline{X}$. The map $\beta_\partial : \partial\overline{X} \times_0 \partial\overline{X} \to \partial\overline{X} \times \partial\overline{X}$ is the 'blow-down map' for this resolution. If (y, y') are local coordinates for $\partial\overline{X} \times \partial\overline{X}$, $r = |y - y'|$, and $\theta = (y - y')/r$, then (r, θ, y) give local coordinates for $\partial\overline{X} \times_0 \partial\overline{X}$, and $\beta_\partial(r, \theta, y) = (y, y + r\theta)$. The kernel of the scattering operator is recovered as an asymptotic limit of the resolvent kernel. Let $\kappa(A)$ denotes the lift of the integral kernel of A (with respect to the measure on $\partial\overline{X}$ induced by the metric $h|_{\partial\overline{X}}$) to $\partial\overline{X} \times_0 \partial\overline{X}$. Then [2, 12]

$$\kappa(S_V(\zeta)) = \beta_\partial^* \left((xx')^{-\zeta} G_\zeta \right) \big|_{T \cap B} \tag{3.1}$$

where $T \cap B$ is the intersection of the top and bottom faces of $\overline{X} \times_0 \overline{X}$. From this formula and Theorem 2.1, we easily obtain:

Theorem 3.1. *The decomposition*

$$\kappa(S_V(\zeta)) = r^{-2\zeta} F_\zeta + \beta_\partial^*(K_\zeta)$$

holds, where F_ζ and K_ζ are meromorphic maps respectively into $C^\infty(\partial\overline{X} \times_0 \partial\overline{X})$ and $C^\infty(\partial\overline{X} \times \partial\overline{X})$, and F_ζ is holomorphic in $\mathbb{C} - \frac{1}{2}\mathbb{Z}$. At poles $\zeta_0 \notin \mathbb{C} - \frac{1}{2}\mathbb{Z}$, the kernel K_ζ has polar part with finite Laurent series and coefficients in $C^\infty(\partial\overline{X} \times \partial\overline{X})$. The set of such poles is contained in the set of poles of $R_V(\zeta)$. For $\zeta_0 \in Z_p$ with $\zeta_0 \notin \frac{1}{2}n + \mathbb{Z}$, $S_V(\zeta)$ has a semi-simple pole at ζ_0 whose residue has rank m_{ζ_0}.

Note that the distribution $r^{-2\zeta}$ has poles at $\zeta \in \frac{1}{2}n+\mathbb{N}$, giving rise to infinite-rank first-order poles of $S_V(\zeta)$. The statement about poles at $\zeta_0 \in Z_p$ follows from the facts that the resolvent has a semi-simple pole at each such ζ_0 and that the residue is a finite-rank projection onto eigenfunctions of the form $x^{\zeta_0}\psi$ for $\psi \in C^\infty(\overline{X})$ with $\psi|_{\partial\overline{X}} \neq 0$. Thus from (3.1) it follows that $S_V(\zeta)$ has a semi-simple pole with residue $\sum_{i=1}^{m_{\zeta_0}} \langle \varphi_i, \cdot \rangle \varphi_i$ where $\varphi_i = x^{-\zeta_0}\psi_i\big|_{\partial X}$.

It follows from Theorem 3.1 that the scattering operator may be factored as

$$S_V(\zeta) = P(\zeta)(I + K_V(\zeta))Q(\zeta)$$

where $P(\zeta)$ and $Q(\zeta)$ are holomorphic families of invertible elliptic operators in $\mathbb{C} - \frac{1}{2}\mathbb{Z}$, and $K_V(\zeta)$ is a meromorphic compact operator-valued function in $\mathbb{C} - \frac{1}{2}(n+\mathbb{Z})$ with finite polar parts and finite-rank residues at each pole $\zeta_0 \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$. For any $\zeta_0 \notin \frac{1}{2}\mathbb{Z}$,

$$\begin{aligned} \nu_{\zeta_0} &= \operatorname{Tr}\left(\frac{1}{2\pi i}\int_{\gamma_{\zeta_0,\varepsilon}} S_V(\zeta)^{-1} S_V'(\zeta)\, d\zeta\right) \\ &= \operatorname{Tr}\left(\frac{1}{2\pi i}\int_{\gamma_{\zeta_0,\varepsilon}} (I + K_V(\zeta))^{-1} K_V'(\zeta)\, d\zeta\right); \end{aligned}$$

since $\zeta \mapsto (I + K_V(\zeta))$ is a meromorphic family of Fredholm operators, it follows from [4] that ν_{ζ_0} is an integer which may be thought of roughly as the number of zeros minus the number of poles of $(I + K_V(\zeta))$ at $\zeta = \zeta_0$. It may be computed as follows. Near a pole $\zeta_0 \notin \frac{1}{2}(n - \mathbb{N})$, the decomposition

$$(I + K_V(\zeta)) = E(\zeta)\left[\sum_\ell (\zeta - \zeta_0)^{\nu_\ell} P_\ell + H(\zeta)\right] F(\zeta)$$

for holomorphically invertible operator-valued functions $E(\zeta)$ and $F(\zeta)$, finite-rank projections P_ℓ with $P_\ell P_m = \delta_{\ell m}$, integers ν_ℓ, and a holomorphic operator-valued function $H(\zeta)$; moreover a similar decomposition holds for $(I + K_V(\zeta))^{-1}$ with the signs of ν_ℓ reversed [4]. Note that the sum over ℓ is finite and that some ν_ℓ may be negative. If $m_\ell = \operatorname{Rank}(P_\ell)$ then $\nu_{\zeta_0} = \sum_\ell \nu_\ell m_\ell$. From the functional equation $S_V(\zeta)S_V(n - \zeta) = I$, it follows that there are holomorphically invertible operator-valued functions $U(\zeta)$ and $V(\zeta)$ in $\zeta \notin \frac{1}{2}\mathbb{Z}$ with

$$(I + K_V(n - \zeta))^{-1} = U(\zeta)\,(I + K_V(\zeta))V(\zeta)$$

so that the relationship

$$\nu_{\zeta_0} = -\nu_{n-\zeta_0} \tag{3.2}$$

holds.

Note that if $n - \zeta_0 \in Z_p$, then the integer ν_{ζ_0} will be determined both by the poles of $(I + K_V(\zeta))$ and those of $(I + K_V(n - \zeta))$, which appear as zeros of $(I + K_V(\zeta))$ at $\zeta = \zeta_0$.

A careful analysis of the resolvent parametrix construction (see for example [2]) shows that the map

$$(\mathbb{C} - \frac{1}{2}\mathbb{Z}) \times \dot{C}^\infty(X) \ni (\zeta, V) \longmapsto K_V(\zeta)$$

is a continuous mapping away from poles of $K_V(\zeta)$.

Suppose now that $\zeta_0 \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$ is a *simple* resonance of $R_V(\zeta)$, $\zeta_0 \notin Z_p$. The polar part of $R_V(\zeta)$ is

$$(\zeta(n-\zeta) - \zeta_0(n-\zeta_0))^{-1} \langle \psi, \cdot \rangle \, \psi$$

where $\psi \in x^\zeta C^\infty(\overline{X})$ solves the eigenvalue equation $(\Delta_g + V - \zeta_0(n-\zeta_0))\psi = 0$. We claim that $\varphi = x^{-\zeta}\psi\big|_{\partial\overline{X}}$ is a nonzero element of $C^\infty(\partial\overline{X})$. If not, then introducing local coordinates (x, y) on X with $x = x$, we have $\psi \in x^{\zeta+1}C^\infty$ and a power series argument using the eigenvalue equation shows that, in fact, $\psi \in \dot{C}^\infty(X)$, so that ψ is an L^2-eigenfunction, a contradiction. It now follows from (3.1) that $S_V(\zeta)$ has a first-order pole at $\zeta = \zeta_0$ with rank-one residue $(\varphi, \cdot)\varphi$, where $(\,\cdot\,,\,\cdot\,)$ is the real inner product on functions with Riemannian measure induced by the metric $h|_{\partial\overline{X}}$. From the holomorphic invertibility of $P(\zeta)$ and $Q(\zeta)$ near $\zeta = \zeta_0$, it follows that $K(\zeta)$ has a simple pole at $\zeta = \zeta_0$. Thus:

Proposition 3.2. *If $R_V(\zeta)$ has a simple resonance at $\zeta_0 \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$, then $S_V(\zeta)$ also has a simple resonance at ζ_0. If $R_V(\zeta)$ is holomorphic at $\zeta_0 \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$, then so is $S_V(\zeta)$.*

Remark 3.3. If (X, g) has constant curvature near infinity, Proposition 3.2 holds for any simple resolvent resonance with $\zeta \notin n/2 - \mathbb{N}$ (i.e., including any simple poles at points $\zeta \in \mathbb{Z}$; later, we will analyze $\zeta \in n/2 - \mathbb{N}$ separately). The only detail to check is the argument that shows that $\varphi = x^{-\zeta}\psi\big|_{\partial\overline{X}}$ is a nonzero element of $C^\infty(\partial\overline{X})$. Since ζ_0 and $(n - \zeta_0)$ differ by an integer, one can no longer show that $\psi \in \dot{C}^\infty(X)$ by a formal power series argument, but one can show that if $\varphi = 0$, then $\psi \in x^n C^\infty(\overline{X})$, which is sufficient to show that $\psi \in L^2(X)$ and derive a contradiction.

Note that, if *all* resonances of $R_V(\zeta)$ were simple, Proposition 3.2 and the conclusion of Theorem 3.1 that the set of scattering poles is contained in the set of resolvent resonances would imply equality of these two sets away from $\frac{1}{2}(n - \mathbb{N})$ and those ζ with $n - \zeta \in Z_p$. In the next two sections we shall show that all resonances are simple for 'generic' V. The following continuity result for scattering poles will be useful.

Proposition 3.4. *Let $\zeta_0 \in \mathcal{S}$ with $\zeta_0 \in \mathbb{C} - \frac{1}{2}(n - \mathbb{N})$. There is a $\delta > 0$ so that for all $V \in \dot{C}^\infty(X)$ with $d(V, 0) < \delta$ and that some $\varepsilon > 0$, the projection*

$$\nu_{\zeta_0}(V) = \operatorname{Tr}\left(\frac{1}{2\pi i} \int_{\gamma_{\zeta_0,\varepsilon}} S_V^{-1}(\zeta) S_V'(\zeta) \, d\zeta \right)$$

is continuous as a map from $\dot{C}^\infty(X)$ to $\mathbb{Z}$. In particular, $\nu_{\zeta_0}(V) = \nu_{\zeta_0}(0)$ for such V.

Proof. This is a consequence of the continuity of the map $(\zeta, V) \longmapsto S_V(\zeta)$. □

4. Perturbation Theory of Resonances

Now we apply Agmon's perturbation theory of resonances [1] to study the behavior of resonances under potential perturbations. For a fixed asymptotically hyperbolic manifold (X, g) we consider the family of operators P_V where V ranges over complex-valued $\dot{C}^\infty(X)$ functions. For any such V, Theorem 2.1 guarantees

that $R_V(\zeta) = (P_V - \zeta(n-\zeta))^{-1}$ admits a meromorphic continuation to any half-plane $\Re(\zeta) > n/2 - N$, N a positive integer, as a mapping from $B_0 = x^N L^2(X)$ to $B_1 = x^{-N} L^2(X)$. We will set $\mathcal{R}_V(z) = (P_V - z)^{-1}$ with the understanding that z lies on the second sheet of the Riemann surface for the inverse function of $f(\zeta) = \zeta(n-\zeta)$, so that $\mathcal{R}_V(z)$ is the meromorphic continuation of the resolvent to the second sheet. Using the fact that $P_V : \dot{C}^\infty(X) \to \dot{C}^\infty(X)$ together with Theorem 2.1, it is not difficult to check that the operator P_V satisfies the hypotheses of Agmon's abstract theory.

To study the perturbation of a resonance z_0, Agmon introduces auxiliary operators and Banach spaces associated to an open connected domain Δ containing z_0 with C^1 boundary Γ. Let B_Γ be the subset of B_1 consisting of functions of the form

$$u = f + \int_\Gamma \mathcal{R}_V(w)\, \Phi(w)\, dw$$

where $f \in B_0$ and $\Phi \in C(\Gamma; B_0)$, the continuous functions on Γ with values in B_0. Finally, let Y be the closed subspace of $B_0 \times C(\Gamma, B_0)$ consisting of those (g, Φ) with

$$0 = g + \int_\Gamma \mathcal{R}_V(w)\, \Phi(w)\, dw.$$

The space B_Γ is a Banach space as the quotient of $B_0 \times C(\Gamma, B_0)$ by the closed subspace Y when equipped with the quotient norm

$$\|u\|_{B_\Gamma} = \inf \left\{ \|\, f\,\|_{B_0} + \|\Phi\|_{C(\Gamma; B_0)} : u = f + \int_\Gamma \mathcal{R}_V(w)\, \Phi(w)\, dw \right\}.$$

The space B_Γ satisfies $B_0 \subset B_\Gamma \subset B_1$, where the canonical injections are continuous.

The theory of [1] implies that there is a closed operator $P_V^\Gamma : \mathcal{D}(P_V^\Gamma) \to B_\Gamma$ which is a restriction of P_V is a sense we will make precise, and whose eigenvalues in Δ are exactly the resonances of $\mathcal{R}_V(z)$ in Δ. In fact, let $\overline{P}_V$ be the closure of P_V as a densely defined operator from B_1 to itself. Then $P_V^\Gamma u = \overline{P}_V u$ for all $u \in \mathcal{D}(P_V^\Gamma)$. The Laurent expansion of $\mathcal{R}_V^\Gamma(z) = (P_V^\Gamma - z)^{-1}$ near a resonance $z_0 \in \Delta$ takes the form

$$\sum_{j=-k}^{-1} (z - z_0)^j A_j^\Gamma + H^\Gamma(z)$$

where $H^\Gamma(z)$ is a holomorphic $\mathcal{L}(B_\Gamma)$-valued function in a neighborhood of z_0 and the A_j^Γ are finite-rank operators belonging to $\mathcal{L}(B_\Gamma, \mathcal{D}(P_V^\Gamma))$. For $f \in B_0$ we have

$$A_j^\Gamma f = A_j f$$

where A_j are the corresponding Laurent coefficients for $\mathcal{R}_V(z)$.

Now fix $V \in \dot{C}^\infty(X)$ and set $P(t) = P_V + tW$ for another potential $W \in \dot{C}^\infty(X)$. The operators $P(t)$ have resolvents which admit meromorphic continuation to $\mathcal{L}(B_0, B_1)$-valued meromorphic functions in $\Re(s) > n/2 - N$ for any fixed $N > 0$. Moreover, for t small and a fixed region Δ, the spaces $B_\Gamma(t)$ corresponding to $P(t)$ are equal as sets and carry equivalent norms, and the operators $P_\Gamma(t)$ form an analytic family of type (A) in the sense of Kato [13]. Let $\mathcal{R}(t,z) = (P(t) - z)^{-1}$ and $\mathcal{R}^\Gamma(t,z) = (P^\Gamma(t) - z)^{-1}$. Theorem 7.7 of [1] shows that, for small t, the resolvents $\mathcal{R}^\Gamma(t,z)$ and $\mathcal{R}(t,z)$ coincide on B_0, possess the same set of poles for each fixed t,

and $\operatorname{Ran}(A_j^\Gamma(t)) = \operatorname{Ran}(A_j(t))$, where $A_j(t)$ and $A_j^\Gamma(t)$ are the respective Laurent coefficients of $\mathcal{R}(t,z)$ and $\mathcal{R}^\Gamma(t,z)$ at a given pole in Δ.

5. Generic Simplicity of Resonances

For L^2 eigenvalues of the Laplacian and its perturbations, it has long been known that 'generic' potential perturbations split degenerate eigenvalues so that a single eigenvalue of multiplicity m becomes m simple eigenvalues, localized near the original eigenvalue (see for example [39], where generic simplicity is proved for the Laplacian on compact manifolds, and Kato [13] for the background in perturbation theory of linear operators; Uhlenbeck's methods adapt without difficulty to eigenvalues below the continuous spectrum). The purpose of this section is to show that the same is true of the resolvent resonances.

Theorem 5.1. *The set E of potentials $V \in \dot{C}^\infty(X)$ for which all eigenvalues of $\Delta_g + V$ and all resonances of $\Delta_g + V$ in $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$ are simple is open and dense in $\dot{C}^\infty(X)$.*

We will follow rather closely the argument of [14] except that Agmon's perturbation theory replaces the exterior complex scaling used there. Since generic simplicity results for eigenvalues are well-known we will only prove the genericity for the resonances.

For positive integers N and real numbers $r > 0$, we define

$$\mathcal{R}_N^r = \left\{ \zeta \in \mathcal{R} : |\zeta| < r, \ \operatorname{dist}(\zeta, \frac{1}{2}(n - \mathbb{N})) > 1/N \right\}$$

and let

$$E_N^r = \left\{ V \in \dot{C}^\infty(X) : \text{each } \zeta \text{ in } \mathcal{R}_N^r \text{ is simple} \right\}.$$

We set

$$E = \cap_{n=1}^\infty \cap_{N=1}^\infty E_N^n,$$

and we define

$$F = \dot{C}^\infty(X) \backslash E.$$

We wish to show that E is dense in $\dot{C}^\infty(X)$, i.e., that F has empty interior. By the Baire category theorem, it suffices to show that $F_N^n = \dot{C}^\infty(X) \backslash E_N^n$ is nowhere dense for each n and N. By the discreteness of the resonance set in $\mathbb{C} - \frac{1}{2}(n - \mathbb{N})$, it suffices to show that for any $V \in F_N^n$, any non-simple pole ζ_0, and any $\varepsilon > 0$, there is a W with $x(W,0) < \varepsilon$ so that $V + W$ has only simple poles in a neighborhood of ζ_0. As in Section 4, it will be convenient to work with the meromorphically continued resolvent $\mathcal{R}(z)$, and we will denote by z_0 the point on the second sheet of the Riemann surface for $\mathcal{R}(z)$ corresponding to ζ_0.

Consider the family of operators $P_{V+W} = \Delta_g + V + W$, where $d(W,0) < \varepsilon_0$, and $\varepsilon_0 > 0$ is to be chosen (here $d(\cdot,\cdot)$ is the metric on $\dot{C}^\infty(X)$ defined in (1.5)). Let Γ be a contour enclosing z_0 and no other pole of $\mathcal{R}(z)$. For ε_0 small enough, the operators P_{V+W}^Γ defined in Agmon's abstract theory can be considered to act on a single Banach space B_Γ, and the associated projection

$$\Pi_{V+W}^\Gamma = \frac{1}{2\pi i} \int_\Gamma (P_{V+W}^\Gamma - w)^{-1} \, dw$$

is analytic in W with $x(W,0) < \varepsilon_0$, and of constant rank, say m. As in [14], we note that either

(1) For each $\varepsilon > 0$, there is a W with $x(W,0) < \varepsilon$ so that P^{Γ}_{V+W} has at least two distinct eigenvalues, or
(2) There is an $\varepsilon > 0$ so that for all W with $x(W,0) < \varepsilon$, P^{Γ}_{V+W} has a single eigenvalue $z(W)$ and there is an integer $k(W)$, $1 \leq k(W) \leq m$, so that
$$(P^{\Gamma}_{V+W} - z(W))^{k(W)}\Pi^{\Gamma}_{V+W} = 0 \quad (P^{\Gamma}_{V+W} - z(W))^{k(W)-1}\Pi^{\Gamma}_{V+W} \neq 0\,.$$

If case (2) does not occur, we can split resonances repeatedly by small perturbations. Thus we will suppose that case (2) does occur and obtain a contradiction.

First, note that $k(W)$ is locally constant so by taking ε_0 small enough we may assume that $k(W)$ is constant for W with $x(W,0) < \varepsilon_0$. As in [14] we consider in turn the possibilities $k(W) = 1$ (the semi-simple case) and $k(W) \geq 2$.

First suppose that $k(W) = 1$, that $z(W)$ is an eigenvalue of P^{Γ}_{V+W} of multiplicity m, and let $\{\psi_i\}_{i=1}^m$ be a basis for $\mathrm{Ran}(A^{\Gamma}_{-1})$, where A^{Γ}_{-1} occurs in the Laurent expansion for $(P^{\Gamma}_V - z)^{-1}$ at $z = z_0$. The vectors $\{\psi_j\}_{j=1}^m$ belong to $B_\Gamma \subset B_1$ and may be chosen to diagonalize A^{Γ}_{-1} as in Proposition 2.4. Let $\{f_j\}_{j=1}^m$ be a set of vectors in B_0 with $\langle \psi_i, f_j\rangle = \delta_{ij}$. Finally, for fixed W, let $L(t) = P^{\Gamma}_{V+tW}$, let $\Pi_t = \Pi^{\Gamma}_{V+tW}$, let $\psi_i(t) = \Pi_t\psi_i$, and let $z(t) = z(tW)$. By differentiating the eigenvalue equation
$$(L(t) - z(t))\psi_i(t) = 0$$
at $t = 0$, we recover the identity
$$(W - z'(0))\psi_i + (L(0) - z(0))\psi_i'(0) = 0.$$
We now apply the projection Π_0 to both sides, pair with f_j, and use the fact that $(L(0) - z(0))\Pi_0 = \Pi_0(L(0) - z(0)) = 0$ to conclude that
$$\langle f_j, \Pi_0 W\psi_i\rangle = z'(0)\delta_{ij}$$
From the choice of $\{f_i\}$ and the fact that $\Pi_0 = \sum_i \langle \psi_i, \cdot\,\rangle\, \psi_i$ it now follows that
$$\langle \psi_i, W\psi_j\rangle = z'(0)\delta_{ij}$$
Since this must hold for any $W \in \dot{C}^\infty(X)$ (in particular for all $W \in C_0^\infty(U)$ with U an open subset of X), it follows that at least one of the ψ_i vanishes on U, and hence on X by unique continuation. This gives a contradiction.

Now suppose that $z(W)$ is not semi-simple, but that there is a fixed $k \geq 2$ so that
$$(L(t) - z(t))^k\Pi_t = 0, \quad (L(t) - z(t))^{k-1}\Pi_t \neq 0.$$
Choose a vector $h \in B$ with $\psi(t) = (L(t) - z(t))^{k-1}\Pi_t h \neq 0$, so that $(L(t) - z(t))\psi(t) = 0$. Let $\psi = \psi(0)$. A perturbation calculation again leads to
$$(W - z'(0))\psi + (L - z(0))\psi'(0) = 0. \tag{5.1}$$
If we apply the projection Π_0 to both sides of (5.1) and pair with a vector $f \in B_0$ with $\Pi_0 f = \psi$, we obtain
$$\langle \psi, W\psi\rangle = z'(0)\,\langle f, \Pi_0\psi\rangle\,.$$

We have used the fact that Π_0 is symmetric with respect to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$. To evaluate the right-hand side, we use the fact that $L - z(0)$ preserves B_0 to write

$$\begin{aligned}\langle f, \Pi_0\psi\rangle &= \left\langle f, \Pi_0(L-z(0))^{k-1}\Pi_0 h\right\rangle \\ &= \left\langle \psi, \Pi_0(L-z(0)^{k-1})\Pi_0 h\right\rangle \\ &= \left\langle (L-z(0))^{k-1}\Pi_0 h, (L-z(0)^{k-1})\Pi_0 h\right\rangle \\ &= \left\langle (L-z(0))^{k-2}\Pi_0 h, (L-z(0)^{k})\Pi_0 h\right\rangle \\ &= 0\end{aligned}$$

so that $\langle\psi, W\psi\rangle = 0$ for all $W \in \dot{C}^\infty(X)$. It follows that ψ vanishes on X, a contradiction.

We have now shown that for each $\varepsilon > 0$, there is a W with $d(W,0) < \varepsilon$ so that P^Γ_{V+W} has at least two distinct eigenvalues. It follows that any resonance can be split by an arbitrarily small perturbation $W \in \dot{C}^\infty(X)$. This fact implies that the set E of potentials V for which $\Delta_g + V$ has only simple resonances in $\mathbb{C} - \frac{1}{2}(n-\mathbb{N})$ is open and dense in $\dot{C}^\infty(X)$, and Theorem 5.1 is proved.

In case (X,g) has constant curvature near infinity, this result can be improved if we work with the class $C_0^\infty(U)$ for a fixed open subset U of X. In this case the methods of [6] can be used to show that the resolvent of $\Delta_g + V$ has a meromorphic continuation with only finite-rank poles, including any poles at $\zeta_0 \in \frac{1}{2}(n-\mathbb{N})$. One can then apply the above arguments without essential changes to prove:

Theorem 5.2. *Let U be a fixed open subset of X with compact closure. The set E of potentials $V \in C_0^\infty(U)$ for which all eigenvalues and all resonances of $\Delta_g + V$ are simple is open and dense in $C_0^\infty(U)$.*

6. Resolvent Resonances and Scattering Poles

Finally, we give the proofs of Theorems 1.1 and 1.2.

To prove Theorem 1.1, we choose a $\dot{C}^\infty(X)$ potential V so that all of the eigenvalues and resonances of $\Delta_g + V$ are simple. We further choose V small enough that, for a given point $(\zeta_0, m_{\zeta_0}) \in \mathcal{R}$ with $\zeta_0 \notin \mathbb{C} - \frac{1}{2}(n-\mathbb{N})$, some $\varepsilon > 0$, and any $t \in (0,2)$, no resonances of $\Delta_g + tV$ cross the circle $\gamma_{\zeta_0,\varepsilon}$ of radius ε about ζ_0, and the projection

$$\Pi_{tV} = \frac{1}{2\pi i}\int_{\gamma_{\zeta_0,\varepsilon}} (2\zeta - n)(\Delta_g + tV - \zeta(n-\zeta))^{-1}\, d\zeta$$

(as well as its analogue for $n - \zeta_0$ if $\zeta_0 \in Z_p$) is continuous in t. It follows from Kato-Rellich perturbation theory for small t, the rank of Π_{tV} is continuous, so that

$$m(t) = \operatorname{rank}(\Pi_{tV}) = m_{\zeta_0}$$

is constant for t small. On the other hand, for $t \neq 0$ and small, the resonances of $\Delta_g + V$ are simple, and the scattering operator has m_{ζ_0} simple poles near ζ_0 by Proposition 3.2, and $m_{n-\zeta_0}$ simple zeros by the remarks following (1.4) and the fact that an eigenvalue of multiplicity $m_{n-\zeta_0}$ splits into $m_{n-\zeta_0}$ simple eigenvalues near $n - \zeta_0$ under perturbation. Finally, by Proposition 3.4, $m_{\zeta_0} - m_{n-\zeta_0} = \nu_{\zeta_0}$, where ν_{ζ_0} is the multiplicity of the scattering pole ζ_0 of $S(\zeta)$, the scattering operator for Δ_g. Hence $\zeta_0 \in \mathcal{S}$, and $m_{\zeta_0} - m_{n-\zeta_0} = \nu_{\zeta_0}$. Theorem 1.1 is proved.

For poles $\zeta_0 \in \mathbb{C} - \frac{1}{2}(n-\mathbb{N})$, the proof breaks down for several reasons: (i) the resolvent may have infinite rank poles at these points, (ii) the scattering operator

may have infinite rank zeros at the points $\zeta \in n/2 - \mathbb{N}$, and (iii) the proof of Proposition 3.2 breaks down. If (X, g) has constant curvature in a neighborhood of infinity, the dimension of X is even, and the potential perturbation V is compactly supported in X, we can show that (i) any resolvent resonances for P_V occuring at points $\zeta \notin n/2 - \mathbb{N}$ are semi-simple, and so can be perturbed to simple resonances under small potential perturbations, and (ii) the resolvent is actually holomorphic at $\zeta \in n/2 - \mathbb{N}$, and the scattering operator $S_V(\zeta)$ vanishes at these points. These observations will enable us to prove Theorem 1.2.

Following [30], we define a renormalized scattering operator by setting

$$S_{r,V}(\zeta) = 2^{2\zeta - n} \frac{\Gamma(\zeta - n/2)}{\Gamma(n/2 - \zeta)} S_V(\zeta). \tag{6.1}$$

It follows from [12] that $S_{r,V}(\zeta)$ admits the factorization

$$S_{r,V}(\zeta) = P(\zeta)(I + K_V(\zeta))P(\zeta)$$

where now $P(\zeta)$ is a holomorphic family of invertible operators with $P(\zeta)P(n-\zeta) = I$, and $K_V(\zeta)$ is a meromorphic family of compact operators with finite polar parts whose Laurent coefficients are finite-rank operators. Moreover the map $(\zeta, V) \longmapsto K_V(\zeta)$ is continuous away from poles of $K_V(\zeta)$. We now define $\mathcal{S}$ to be the set of poles of $(I + K_0(\zeta))$ with multiplicities

$$\nu_{\zeta_0} = \mathrm{Tr}\left(\frac{1}{2\pi i} \int_{\gamma_{\zeta_0,\varepsilon}} S_{r,0}(\zeta)^{-1} S'_{r,0}(\zeta)\, d\zeta \right).$$

This definition coincides with the previous definition for all $\zeta_0 \in \mathbb{C} - \frac{1}{2}(n - \mathbb{N})$ and is well-defined for any ζ. Note that the distribution kernel of $S_{r,V}(\zeta)$ is recovered from the Schwarz kernel of the resolvent by (compare (3.1))

$$\kappa(S_V(\zeta)) = 2^{2\zeta - n} \frac{\Gamma(\zeta - n/2)}{\Gamma(n/2 - \zeta)}\, \beta^* \left((xx')^{-\zeta} G_\zeta \right)\big|_{T \cap B}$$

so that for $\zeta_0 \notin \frac{1}{2}n - \mathbb{N}$, Remark 3.3 shows that if $R_V(\zeta)$ has a simple pole at ζ_0, then $S_V(\zeta)$ does also. It remains to analyze what happens for the points $\zeta_0 \in \frac{1}{2}n - \mathbb{N}$ and it is here that our assumption on even dimension enters. In this case, we can prove:

Lemma 6.1. *Suppose that X has constant curvature in a neighborhood of infinity, V is compactly supported, and $\dim X$ is even. Then the resolvent $R_V(\zeta)$ is holomorphic at $\zeta \in \frac{1}{2}n - \mathbb{N}$, and $S_V(\zeta)$ vanishes at these points.*

Proof. This proof is inspired by the proof of [7], Lemma 2.8, and generalizes its argument. Let $G_{0,\zeta}(w, w')$ be the integral kernel of $R_0(\zeta)$ with respect to Riemannian measure on $\mathbb{H}^{n+1}$, where $w = (y, x)$, $w' = (y', x')$ for $y, y;' \in \mathbb{R}^n$ and $x, x' > 0$ (upper half-space model). Then $G_{0,\frac{1}{2}n-k} \in (xx')^{\frac{1}{2}n+k} C^\infty(\mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \setminus \Delta)$ where Δ denotes the diagonal. This may be checked by explicit computation of the resolvent or by using the formula

$$G_{0,\zeta}(w, w') - G_{0,n-\zeta}(w, w') = \int_{\mathbb{R}^n} e_{0,\zeta}(w, y'') e_{0,n-\zeta}(w, y'')\, dy''$$

where

$$e_{0,\zeta}(w, y'') = \pi^{-n/2} \frac{\Gamma(\zeta)}{\Gamma(\zeta - n/2)} \frac{x^\zeta}{(|y - y''|^2 + x^2)^\zeta}.$$

The point is that $e_{0,\zeta}$ vanishes at $s \in \frac{1}{2}n - \mathbb{N}$ if $\dim X = n+1$ is even. It follows from the iterative parametrix construction in [6] that the meromorphically continued resolvent kernel of $R(\zeta)$ has no pole at $\zeta = n/2 - k$, since its residue would correspond to a solution of the eigenvalue equation with $y^{n/2+k}$ (hence square-integrable) behavior at infinity. It further follows that the resolvent kernel belongs to $(xx')^{n/2+k}C^\infty(X \times X)$ away from the diagonal, so that the Schwartz kernel of the scattering operator at $\zeta = n/2 - k$, obtained as a boundary value of the resolvent rescaled by $(xx')^{\frac{1}{2}(k-n)}$, is zero, and hence the scattering operator is zero. $\square$

From the continuity of the map $(\zeta, V) \longmapsto K_V(\zeta)$ it follows that the integer-valued function

$$\nu_{\zeta_0}(V) = \operatorname{Tr}\left(\frac{1}{2\pi i}\int_{\gamma_{\zeta_0,\varepsilon}} S_{r,V}(\zeta)^{-1} S'_{r,V}(\zeta)\, d\zeta\right)$$

is constant for some $\delta > 0$ and all $V \in \dot{C}^\infty(X)$ with $d(0,V) < \delta$. The continuity of the projection Π_V for resolvent poles can also be established for poles in $\frac{1}{2}(n - \mathbb{N})$ since any such poles are known to have finite rank. We are now able to argue as before to complete the proof.

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